

QUANTIZATION OF SOME MODULI SPACES OF PARABOLIC VECTOR BUNDLES ON \mathbb{CP}^1

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ABSTRACT. We address quantization of the natural symplectic structure on a moduli space of parabolic vector bundles of parabolic degree zero over \mathbb{CP}^1 with four parabolic points and parabolic weights in $\{0, 1/2\}$. Identifying such parabolic bundles as vector bundles on an elliptic curve, we obtain explicit expressions for the corresponding non-abelian theta functions. These non-abelian theta functions are described in terms of certain naturally defined distributions on the compact group $SU(2)$.

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1. INTRODUCTION

Let X be a compact connected Riemann surface, or equivalently a smooth complex projective curve. It is well known that the moduli spaces of vector bundles over X have a canonical symplectic structure [Go], with integral symplectic form. Indeed, being naturally identified with spaces of flat connections on a compact oriented surface, these are important classical phase spaces of Chern-Simons theory. The natural question of their quantization was addressed in many articles [Hi], [AdPW].

The geometric quantization of moduli spaces \mathcal{N} of vector bundles over X in a so-called Kähler polarization leads to what is known as spaces of non-abelian theta functions. More concretely, the Kähler polarized Hilbert spaces, at level $k = 1, 2, \dots$, are the spaces

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$H^0(\mathcal{N}, \mathcal{L}^k)$, where \mathcal{L} is a determinant line bundle, endowed with a natural Chern connection, whose curvature coincides with the symplectic form. A projectively flat connection was constructed by Hitchin on the space of complex structures on \mathcal{N} [Hi], providing a way of identifying different choices of Kähler polarized Hilbert spaces.

However, an explicit identification between Kähler polarized quantizations and real polarized ones has only been found in a few examples, notably the case when X is an elliptic curve [AdPW, FMN2], using the relationship between the moduli spaces in this case and a certain class of abelian varieties. In turn, a comparison between real and Kähler quantizations for abelian varieties was obtained using a coherent state transform [FMN2, FMN1, BMN].

In this article, we follow the analogous geometric quantization program for the moduli space $\mathcal{M}_P(r)$ of parabolic bundles of rank r over \mathbb{CP}^1 with four parabolic points and parabolic weights in $\{0, 1/2\}$ with parabolic degree zero [MS, MY]. It is known that, as in the case of vector bundles, there is a determinant line bundle ζ_P over the moduli space of parabolic bundles endowed with a natural Chern connection, whose curvature is the (generally singular) Kähler form.

Let X be the elliptic curve which has a degree two map to \mathbb{CP}^1 ramified over the parabolic points. Using the description of the parabolic bundles of above type as holomorphic vector bundles over X equipped with a lift of the involution corresponding to the degree two covering [Bi1], we see that, for a given choice of parabolic structures on these 4 points, $\mathcal{M}_P(r)$ has dimension $d \leq r/2$ and we have a canonical isomorphism

$$\mathcal{M}_P(r) \cong X^d / \Gamma_d \cong \mathbb{CP}^d,$$

where Γ_d is the semi-direct product $(\mathbb{Z}/2\mathbb{Z})^d \rtimes \Sigma_d$ for the natural action on $(\mathbb{Z}/2\mathbb{Z})^d$ of the symmetric group Σ_d for d elements. Moreover, for the natural polarization line bundle L on the abelian variety X^d (associated to a Kähler form of area one on X), we obtain an isomorphism $\phi^* \zeta_P \cong L^2$, where $\zeta_P \rightarrow \mathcal{M}_P(r)$ is the determinant line bundle and $\phi : X^d \rightarrow \mathcal{M}_P(r)$ is the natural quotient (see Sections 2 and 3).

This very concrete description allows the expression of the quantization Hilbert space at level k , namely $H^0(\mathcal{M}_P, \zeta_P^k)$, in terms of the (abelian) theta functions of level $2k$ on X^d , and the comparison of real and Kähler polarized Hilbert spaces. For this, we need to apply the framework of [FMN2] for non-abelian theta functions over the moduli space of rank 2 vector bundles, with trivial determinant, over X (see Section 4). The so-called coherent state transform for Lie groups [Ha], is an analytic tool which, given an invariant Laplacian on a compact Lie group K , associates holomorphic functions on the complexification $K_{\mathbb{C}}$ to square integrable functions on K . This set up can be extended to appropriate spaces of distributions on K [FMN1]. Non-abelian theta functions of level k on \mathcal{M}_P are then described in terms of Ad -invariant holomorphic functions on the group $SL(2, \mathbb{C})$ with special quasi-periodicity properties. These holomorphic functions are obtained from elements in a vector space of distributions on the compact real form $SU(2)$ by applying the coherent state transform, for time $1/(k+2)$ (see Theorem 4.7).

2. A MODULI SPACE OF PARABOLIC VECTOR BUNDLES OVER \mathbb{CP}^1

Fix a point $p_0 \in \mathbb{CP}^1 \setminus \{0, 1, \infty\}$. Consider the divisor

$$(2.1) \quad S := \{0, 1, \infty, p_0\} \subset \mathbb{CP}^1.$$

Let

$$(2.2) \quad f : X \longrightarrow \mathbb{CP}^1$$

be the unique double cover ramified exactly over D . Therefore, X is a complex elliptic curve. Let $\text{Pic}^0(X)$ be the moduli space of topologically trivial holomorphic vector bundles on X .

Lemma 2.1. *Any polystable vector bundle E over X of rank r and degree zero is isomorphic to a direct sum $\bigoplus_{i=1}^r L_i$, where $L_i \in \text{Pic}^0(X)$.*

The isomorphism classes of line bundles L_i , $1 \leq i \leq r$, are uniquely determined by E up to a permutation of $\{1, \dots, r\}$.

Proof. The first statement follows immediately from Atiyah's classification of holomorphic vector bundles on X (see [At2]). This also follows from the facts that E is given by a representation of the abelian group $\pi_1(X)$ in $\text{U}(r)$ [NaSe].

The uniqueness of L_i up to a permutation of $\{1, \dots, r\}$ follows immediately from [At1, p. 315, Theorem 2(ii)]. \square

We will consider parabolic vector bundles over \mathbb{CP}^1 with S (see (2.1)) as the parabolic divisor. Let E be a holomorphic vector bundle on \mathbb{CP}^1 . A *quasi-parabolic* structure on E is a filtration of subspaces

$$E_y =: F_{y,1} \supsetneq \cdots \supsetneq F_{y,j} \supsetneq \cdots \supsetneq F_{y,a_y} \supsetneq F_{y,a_y+1} = 0$$

over each point $y \in S$. A *parabolic* structure on E is a quasi-parabolic structure as above together with real numbers

$$(2.3) \quad 0 \leq \alpha_{y,1} < \cdots < \alpha_{y,j} < \cdots < \alpha_{y,a_y} < 1$$

associated to the quasi-parabolic flags. (See [MS], [MY].) The numbers $\alpha_{y,j}$ in (2.3) are called *parabolic weights*. The *multiplicity* of the parabolic weight $\alpha_{y,j}$ is $\dim_{\mathbb{C}} F_{y,j}/F_{y,j+1}$.

For notational convenience, a parabolic vector bundle $(E, \{F_{y,j}\}, \{\alpha_{y,j}\})$ defined as above will also be denoted by E_* . The *parabolic degree* is defined to be

$$\text{par-deg}(E_*) := \text{degree}(E) + \sum_{y \in S} \sum_{j=1}^{a_y} \alpha_{y,j} \cdot \dim(F_{y,j}/F_{y,j+1}).$$

Fix an integer $r \geq 2$. For each point $y \in S$, fix an integer $m_y \in [0, r]$. Let \mathcal{M}_P be the moduli space of semistable parabolic vector bundles E_* on \mathbb{CP}^1 of rank r , with S as the parabolic divisor, such that the parabolic weights at a parabolic point y are $1/2$ with multiplicity m_y and 0 with multiplicity $r - m_y$, and

$$\text{par-deg}(E_*) = 0.$$

(See [MY] for the construction of \mathcal{M}_P .) The moduli spaces of parabolic bundles are irreducible normal complex projective varieties. We will see later that the above moduli space \mathcal{M}_P is smooth. Note that \mathcal{M}_P is empty if $\sum_{y \in S} m_y$ is an odd integer.

We will assume that $\sum_{y \in S} m_y$ is an even integer.

For any integer $m \geq 1$, we will construct a finite group Γ_m equipped with an action of it on the Cartesian product $\text{Pic}^0(X)^m$.

Let Σ_m be the group of permutations of $\{1, \dots, m\}$. This group acts on the Cartesian product $(\mathbb{Z}/2\mathbb{Z})^m$ by permuting the factors. So any permutation $\tau \in \Sigma_m$ of $\{1, \dots, m\}$ sends any $(z_1, \dots, z_m) \in (\mathbb{Z}/2\mathbb{Z})^m$ to $(z_{\tau^{-1}(1)}, \dots, z_{\tau^{-1}(m)})$. Let

$$(2.4) \quad \Gamma_m := (\mathbb{Z}/2\mathbb{Z})^m \rtimes \Sigma_m$$

be the semi-direct product corresponding to this action. So Γ_m fits in a short exact sequence

$$(2.5) \quad e \longrightarrow (\mathbb{Z}/2\mathbb{Z})^m \longrightarrow \Gamma_m \longrightarrow \Sigma_m \longrightarrow e$$

of groups. We will construct a natural action of Γ_m on $\text{Pic}^0(X)^m$.

Consider the action of group $\mathbb{Z}/2\mathbb{Z}$ on $\text{Pic}^0(X)$ defined by the involution $L \mapsto L^*$. Acting coordinate-wise, it produces an action of $(\mathbb{Z}/2\mathbb{Z})^m$ on $\text{Pic}^0(X)^m$. On the other hand, the permutation group Σ_m acts on $\text{Pic}^0(X)^m$; as before, the action of any $\tau \in \Sigma_m$ sends any $(z_1, \dots, z_m) \in \text{Pic}^0(X)^m$ to $(z_{\tau^{-1}(1)}, \dots, z_{\tau^{-1}(m)})$. These two actions together produce an action of Γ_m (constructed in (2.4)) on $\text{Pic}^0(X)^m$. Let

$$(2.6) \quad \text{Pic}^0(X)^m \longrightarrow \text{Pic}^0(X)^m / \Gamma_m$$

be the quotient for this action. The quotient $\text{Pic}^0(X)^m / (\mathbb{Z}/2\mathbb{Z})^m$ for the subgroup in (2.5) is identified with $(\text{Pic}^0(X) / (\mathbb{Z}/2\mathbb{Z}))^m$. Hence

$$\text{Pic}^0(X)^m / \Gamma_m = \text{Sym}^m(\text{Pic}^0(X) / (\mathbb{Z}/2\mathbb{Z})).$$

Since $\text{Pic}^0(X) / (\mathbb{Z}/2\mathbb{Z}) = \mathbb{CP}^1$, we have

$$\text{Pic}^0(X)^m / \Gamma_m = \text{Sym}^m(\mathbb{CP}^1) = \mathbb{CP}^m.$$

Note that the quotient map in (2.6) factors through the projection

$$\text{Pic}^0(X)^m \longrightarrow \text{Sym}^m(\text{Pic}^0(X)) := \text{Pic}^0(X)^m / \Sigma_m.$$

But the surjective map

$$\text{Sym}^m(\text{Pic}^0(X)) \longrightarrow \text{Pic}^0(X)^m / \Gamma_m$$

in general is not a quotient for a group action because Σ_m is not a normal subgroup of Γ_m .

Proposition 2.2. *Let d be the (complex) dimension of \mathcal{M}_P . Then $d \leq r/2$. If $d > 0$, then the variety \mathcal{M}_P is canonically isomorphic to the quotient $\text{Pic}^0(X)^d / \Gamma_d$ constructed in (2.6).*

Proof. Let

$$(2.7) \quad \sigma : X \longrightarrow X$$

be the unique nontrivial deck transformation for the covering f in (2.2).

Let $E_* \in \mathcal{M}_P$ be a polystable parabolic vector bundle. It corresponds to a unique holomorphic vector bundle $V \longrightarrow X$ equipped with a lift of the involution σ in (2.7) as an isomorphism of order two

$$(2.8) \quad \tilde{\sigma} : V \longrightarrow \sigma^*V$$

of vector bundles [Bi1]; this means that $\tilde{\sigma}$ is a holomorphic isomorphism of vector bundles, and the composition

$$V \xrightarrow{\tilde{\sigma}} \sigma^*V \xrightarrow{\sigma^*\tilde{\sigma}} \sigma^*\sigma^*V = V$$

is the identity map. We have

$$\text{degree}(V) = 0$$

because $\text{par-deg}(E_*) = 0$ [Bi1, p. 318, (3.12)]. The vector bundle V is polystable because E_* is polystable [BBN, pp. 350–351, Theorem 4.3]. Therefore, from Lemma 2.1 we know that

$$(2.9) \quad V = \bigoplus_{i=1}^r L_i,$$

where $L_i \in \text{Pic}^0(X)$. Recall from Lemma 2.1 that the line bundles L_i are uniquely determined up to a permutation.

For any line bundle L on X of degree zero, the line bundle $L \otimes \sigma^*L$ descends to \mathbb{CP}^1 , where σ is defined in (2.7). Since $\text{Pic}^0(\mathbb{CP}^1) = \{\mathcal{O}_{\mathbb{CP}^1}\}$, it follows that

$$(2.10) \quad \sigma^*L = L^*$$

for all $L \in \text{Pic}^0(X)$.

Since V in (2.9) is isomorphic to σ^*V (see (2.8)), using (2.10),

$$(2.11) \quad \bigoplus_{i=1}^r L_i = \bigoplus_{i=1}^r L_i^*.$$

Therefore, all vector bundles on X corresponding to points of \mathcal{M}_P are of the form

$$(2.12) \quad \bigoplus_{i=1}^a (\xi_i \oplus \xi_i^*) \oplus \bigoplus_{j=1}^{r-2a} \eta_j,$$

where η_j are fixed line bundles on X (these line bundles η_j depend on the numbers m_y but are independent of the point of the moduli space \mathcal{M}_P), and the line bundles ξ_i , $1 \leq i \leq a$, move over $\text{Pic}^0(X)$. From (2.11) it follows that

$$(2.13) \quad \eta_j = \eta_j^*$$

for all j . We note that any vector bundle as in (2.12) satisfying (2.13) admits a lift of the involution σ . Indeed, each η_j has a lift because (2.13) holds. Also, $\xi_i \oplus \xi_i^*$ has a natural lift of the involution σ because $\sigma^*\xi_i = \xi_i^*$. Note that the involution of $\xi_i \oplus \xi_i^*$ interchanges the two direct summands.

Hence we get a surjective morphism

$$(2.14) \quad \mathrm{Pic}^0(X)^a \longrightarrow \mathcal{M}_P$$

that sends any (ξ_1, \dots, ξ_a) to

$$\bigoplus_{i=1}^a (\xi_i \oplus \xi_i^*) \oplus \bigoplus_{j=1}^{r-2a} \eta_j.$$

This morphism clearly factors through the quotient $\mathrm{Pic}^0(X)^a/\Gamma_a$ in (2.6).

For a vector bundle

$$W = \bigoplus_{i=1}^a (\xi_i \oplus \xi_i^*),$$

the unordered pairs $\{\xi_i, \xi_i^*\}$ are uniquely determined by W up to a permutation of $\{1, \dots, a\}$ [At1, p. 315, Theorem 2(ii)]. Using this it follows that the above morphism

$$\mathrm{Pic}^0(X)^a/\Gamma_a \longrightarrow \mathcal{M}_P$$

is an isomorphism. This completes the proof of the proposition. \square

3. DETERMINANT LINE BUNDLE AND KÄHLER FORM ON \mathcal{M}_P

Consider the moduli space \mathcal{M}_P of parabolic vector bundles defined in the previous section. It has a natural (possibly singular) Kähler form; this Kähler form will be denoted by ω_P . There is a determinant line bundle

$$(3.1) \quad \zeta \longrightarrow \mathcal{M}_P.$$

This line bundle ζ has a hermitian structure such that the curvature of the corresponding Chern connection coincides with ω_P . (See [BR], [Bi2], [TZ].)

Consider the dimension d in Proposition 2.2. Let Γ_d be the group defined in (2.4). The quotient $\mathrm{Pic}^0(X)^d/\Gamma_d$ is identified with the moduli space \mathcal{M}_P by Proposition 2.2. Let

$$(3.2) \quad \phi : \mathrm{Pic}^0(X)^d \longrightarrow \mathrm{Pic}^0(X)^d/\Gamma_d = \mathcal{M}_P$$

be the morphism in (2.14). Since ϕ is the quotient map for the action of Γ_d on $\mathrm{Pic}^0(X)^d$, the pulled back line bundle $\phi^*\zeta$ is equipped with a lift of the action of Γ_d on $\mathrm{Pic}^0(X)^d$, where ζ is the determinant line bundle in (3.1).

Let

$$(3.3) \quad L_0 := \mathcal{O}_{\mathrm{Pic}^0(X)}(\mathcal{O}_X) \longrightarrow \mathrm{Pic}^0(X)$$

be the holomorphic line bundle of degree one defined by the point of $\mathrm{Pic}^0(X)$ corresponding to the trivial line bundle \mathcal{O}_X on X . For each $i \in [1, d]$, let

$$(3.4) \quad q_i : \mathrm{Pic}^0(X)^d \longrightarrow \mathrm{Pic}^0(X)$$

be the projection to the i -th factor. The action of Σ_d on $\mathrm{Pic}^0(X)^d$ that permutes the factors in the Cartesian product has a natural lift to an action of Σ_d on the line bundle

$$\bigotimes_{i=1}^d q_i^* L_0 \longrightarrow \mathrm{Pic}^0(X)^d,$$

where L_0 is the line bundle in (3.3).

Recall that the group $(\mathbb{Z}/2\mathbb{Z})^d$ acts on $\text{Pic}^0(X)^d$ using the action of $\mathbb{Z}/2\mathbb{Z}$ on $\text{Pic}^0(X)$ given by the involution $L \rightarrow L^*$. Let

$$(3.5) \quad \sigma_d : (\mathbb{Z}/2\mathbb{Z})^d \rightarrow \text{Aut}(\text{Pic}^0(X)^d)$$

be the corresponding homomorphism.

Theorem 3.1. *For any $g \in (\mathbb{Z}/2\mathbb{Z})^d$, there is a canonical isomorphism of holomorphic line bundles*

$$\sigma_d(g)^* \left(\bigotimes_{i=1}^d q_i^* L_0 \right) \xrightarrow{\sim} \bigotimes_{i=1}^d q_i^* L_0,$$

where σ_d is the homomorphism in (3.5), and L_0 is the line bundle in (3.3).

The line bundle $(\bigotimes_{i=1}^d q_i^* L_0)^{\otimes 2}$ has a canonical lift of the action of Γ_d on $\text{Pic}^0(X)^d$.

There is a Γ_d -equivariant isomorphism of line bundles

$$\left(\bigotimes_{i=1}^d q_i^* L_0 \right)^{\otimes 2} \xrightarrow{\sim} \phi^* \zeta,$$

where ϕ and ζ are defined in (3.2) and (3.1) respectively.

Proof. Consider the automorphism of $\text{Pic}^0(\text{Pic}^0(X))$ induced by the involution of $\text{Pic}^0(X)$ defined by $L \mapsto L^*$. It fixes the line bundle L_0 defined in (3.3), because the above involution $L \mapsto L^*$ fixes the point of $\text{Pic}^0(X)$ corresponding to the trivial line bundle \mathcal{O}_X on X . Note that $\text{Pic}^0(\text{Pic}^0(X))$ is identified with $\text{Pic}^0(X)$ by sending any $\xi \in \text{Pic}^0(X)$ to $\mathcal{O}_{\text{Pic}^0(X)}(\xi - \mathcal{O}_X)$. This identification commutes with the involutions. Since the point of $\text{Pic}^0(X)$ corresponding to \mathcal{O}_X is fixed by the involution, it follows that

$$(3.6) \quad \sigma_d(g)^* \left(\bigotimes_{i=1}^d q_i^* L_0 \right) = \bigotimes_{i=1}^d q_i^* L_0$$

for all $g \in (\mathbb{Z}/2\mathbb{Z})^d$. This proves the first statement of the theorem.

Let $z_0 := (\mathcal{O}_X, \dots, \mathcal{O}_X) \in \text{Pic}^0(X)^d$ be the point. Note that $\sigma_d(g)(z_0) = z_0$ for all $g \in (\mathbb{Z}/2\mathbb{Z})^d$. In view of (3.6), there is a unique isomorphism

$$(3.7) \quad \rho : \sigma_d(g)^* \left(\bigotimes_{i=1}^d q_i^* L_0 \right) \rightarrow \bigotimes_{i=1}^d q_i^* L_0$$

which coincides with the identity map of the fiber $(\bigotimes_{i=1}^d q_i^* L_0)_{z_0}$ over the point z_0 .

Consider the action of $(\mathbb{Z}/2\mathbb{Z})^d$ on $\text{Pic}^0(X)^d$ defined by the homomorphism σ_d in (3.5). For any $g \in (\mathbb{Z}/2\mathbb{Z})^d$, there is a canonical lift of the involution $\sigma_d(g)$ of $\text{Pic}^0(X)^d$ to the line bundle

$$\left(\bigotimes_{i=1}^d q_i^* L_0 \right) \otimes \sigma_d(g)^* \left(\bigotimes_{i=1}^d q_i^* L_0 \right).$$

Using the isomorphism ρ in (3.7), these lifts of the involutions $\sigma_d(g)$, $g \in (\mathbb{Z}/2\mathbb{Z})^d$, together produce a lift of the action of $(\mathbb{Z}/2\mathbb{Z})^d$ on $\text{Pic}^0(X)^d$ to the line bundle

$$(\bigotimes_{i=1}^d q_i^* L_0)^{\otimes 2} \longrightarrow \text{Pic}^0(X)^d.$$

We already noted that the action of Σ_d on $\text{Pic}^0(X)^d$ has a natural lift to an action of Σ_d on the line bundle $\bigotimes_{i=1}^d q_i^* L_0$. This lift to $\bigotimes_{i=1}^d q_i^* L_0$ produces a lift to $(\bigotimes_{i=1}^d q_i^* L_0)^{\otimes 2}$ of the action of Σ_d on $\text{Pic}^0(X)^d$. This action of Σ_d on $(\bigotimes_{i=1}^d q_i^* L_0)^{\otimes 2}$ and the action of $(\mathbb{Z}/2\mathbb{Z})^d$ on $(\bigotimes_{i=1}^d q_i^* L_0)^{\otimes 2}$ constructed above together produce a lift to $(\bigotimes_{i=1}^d q_i^* L_0)^{\otimes 2}$ of the action of Γ_d on $\text{Pic}^0(X)^d$. This proves the second statement of the theorem.

Let $\mathcal{N}_X(r)$ denote the moduli space of semistable vector bundles on X of rank r and degree zero. So, $\mathcal{N}_X(r) = \text{Sym}^r(\text{Pic}^0(X)) := \text{Pic}^0(X)^r / \Sigma_r$.

Let

$$\beta : \text{Pic}^0(X)^d \longrightarrow \mathcal{N}_X(r)$$

be the morphism defined by

$$(3.8) \quad (L_1, \dots, L_d) \longmapsto \bigoplus_{i=1}^d (L_i \oplus L_i^*) \oplus \bigoplus_{j=1}^{r-2d} \eta_j$$

(see (2.12)). Let

$$(3.9) \quad \gamma : \mathcal{M}_P \longrightarrow \mathcal{N}_X(r)$$

be the morphism that sends any parabolic vector bundle on \mathbb{CP}^1 to the corresponding vector bundle on X (see the proof of Proposition 2.2). Clearly,

$$(3.10) \quad \beta := \gamma \circ \phi,$$

where ϕ is constructed in (3.2).

Let ζ_r be the determinant line bundle on the moduli space $\mathcal{N}_X(r)$. We will quickly recall the definition/construction of ζ_r . Let

$$(3.11) \quad \mathcal{P} \longrightarrow X \times \text{Pic}^0(X)$$

be a Poincaré line bundle; this means that for each point $\alpha \in \text{Pic}^0(X)$, the restriction $\mathcal{P}|_{X \times \{\alpha\}}$ lies in the isomorphism class of line bundles defined by the point α . Let

$$(3.12) \quad \pi_2 : X \times \text{Pic}^0(X) \longrightarrow \text{Pic}^0(X)$$

be the natural projection. Define the line bundle

$$\mathcal{L} := \left(\bigwedge^{\text{top}} R^0 \pi_{2*} \mathcal{P} \right)^* \otimes \bigwedge^{\text{top}} R^1 \pi_{2*} \mathcal{P} \longrightarrow \text{Pic}^0(X).$$

It can be shown that \mathcal{L} is independent of the choice of the Poincaré line bundle \mathcal{P} . To see this note that any other Poincaré bundle is of the form $\mathcal{P}_1 := \mathcal{P} \otimes \pi_2^* A$, where A is a line bundle on $\text{Pic}^0(X)$. Form the projection formula,

$$\left(\bigwedge^{\text{top}} R^0 \pi_{2*} \mathcal{P} \right)^* \otimes \bigwedge^{\text{top}} R^1 \pi_{2*} \mathcal{P} = \left(\bigwedge^{\text{top}} R^0 \pi_{2*} \mathcal{P}_1 \right)^* \otimes \left(\bigwedge^{\text{top}} R^1 \pi_{2*} \mathcal{P}_1 \right) \otimes A^{\otimes \chi},$$

where χ is the Euler characteristic of degree zero line bundles on X . Since $\chi = 0$, we have $\mathcal{L} = (\bigwedge^{\text{top}} R^0 \pi_{2*} \mathcal{P}_1)^* \otimes \bigwedge^{\text{top}} R^1 \pi_{2*} \mathcal{P}_1$.

We will show that \mathcal{L} coincides with L_0 defined in (3.3). To prove this, we first note that \mathcal{O}_X is the unique line bundle on X such that $\chi(\mathcal{O}_X) = 0 \neq H^0(X, \mathcal{O}_X)$. From this it follows that the point of $\text{Pic}^0(X)$ defined by \mathcal{O}_X is the canonical theta divisor. This immediately implies that \mathcal{L} is canonically identified with L_0 .

For each $i \in [1, r]$, let \bar{q}_i be the projection of $\text{Pic}^0(X)^r$ to the i -th factor. The line bundle

$$(3.13) \quad \bigotimes_{i=1}^r \bar{q}_i^* \mathcal{L} \longrightarrow \text{Pic}^0(X)^r$$

has a natural action of the group Σ_r of permutations of $\{1, \dots, r\}$. Using this action, the line bundle in (3.13) descends to the quotient $\text{Sym}^r(\text{Pic}^0(X))$ of $\text{Pic}^0(X)^r$. This descended line bundle is the determinant line bundle ζ_r on $\mathcal{N}_X(r) = \text{Sym}^r(\text{Pic}^0(X))$.

For the map γ in (3.9), the pullback $\gamma^* \zeta_r$ coincides with the determinant line bundle ζ on \mathcal{M}_P [BR], [Bi2]. Therefore, from (3.10) we get an isomorphism

$$(3.14) \quad \phi^* \zeta \xrightarrow{\sim} \beta^* \zeta_r.$$

Using the fact that each η_j in (3.8) is a fixed line bundle of order two, from the construction of ζ_r described above it is easy to see that $\beta^* \zeta_r$ has a canonical lift of the action of Γ_d on $\text{Pic}^0(X)^d$. As noted earlier, the line bundle $\phi^* \zeta$ is equipped with an action of Γ_d , where ϕ is constructed in (3.2). It is straightforward to check that the isomorphism in (3.14) intertwines the actions of Γ_d .

For any Poincaré line bundle $\mathcal{P} \longrightarrow X \times \text{Pic}^0(X)$ (see (3.11)), the pullback

$$(\text{Id}_X \times \sigma_1)^* \mathcal{P}^* \longrightarrow X \times \text{Pic}^0(X)$$

is also a Poincaré line bundle, where σ_1 is the involution in (3.5) defined by $L \longmapsto L^*$. Therefore,

$$(3.15) \quad \left(\bigwedge^{\text{top}} R^0 \pi_{2*} \mathcal{P}^* \right)^* \otimes \bigwedge^{\text{top}} (R^1 \pi_{2*} \mathcal{P}^*) = \sigma_1^* \mathcal{L},$$

where π_2 is the projection in (3.12). Since $\mathcal{L} = L_0$,

$$(3.16) \quad \sigma_1^* \mathcal{L} = \sigma_1^* L_0 = L_0;$$

the last isomorphism follows from the fact that the point of $\text{Pic}^0(X)$ corresponding to \mathcal{O}_X is fixed by σ_1 (see also (3.3)). Combining (3.15) and (3.16),

$$(3.17) \quad \left(\bigwedge^{\text{top}} R^0 \pi_{2*} \mathcal{P}^* \right)^* \otimes \bigwedge^{\text{top}} (R^1 \pi_{2*} \mathcal{P}^*) = L_0.$$

Using (3.17), from the constructions of the line bundle ζ_r and the morphism β in (3.8) it follows that

$$\beta^* \zeta_r = \left(\bigotimes_{i=1}^d q_i^* L_0 \right)^{\otimes 2}.$$

In the second part of the theorem we constructed an action of the group Γ_d on the line bundle $(\bigotimes_{i=1}^d q_i^* L_0)^{\otimes 2}$. We noted earlier that $\beta^* \zeta_r$ is equipped with a lift of the action of Γ_d on $\text{Pic}^0(X)^d$. The above isomorphism of $\beta^* \zeta_r$ with $(\bigotimes_{i=1}^d q_i^* L_0)^{\otimes 2}$ is Γ_d -equivariant. In view of the fact, noted earlier, that the isomorphism in (3.14) intertwines the actions of Γ_d , this completes the proof of the theorem. \square

There is a unique translation invariant Kähler form h_0 on $\text{Pic}^0(X)$ of total volume one. Let

$$\omega := \sum_{i=1}^d q_i^* h_0$$

be the Kähler form on $\text{Pic}^0(X)^d$, where q_i is the projection in (3.4). As before, the Kähler form on \mathcal{M}_P will be denoted by ω_P .

Proposition 3.2. *For the morphism ϕ in (3.2),*

$$\phi^* \omega_P = 2\omega.$$

Proof. Let ω_r be the Kähler form on the moduli space $\mathcal{N}_X(r)$. For the map γ in (3.9),

$$(3.18) \quad \gamma^* \omega_r = \omega_P$$

(see [BR]).

It can be shown that

$$(3.19) \quad \beta^* \omega_r = 2\omega,$$

where β is constructed in (3.8). To prove (3.19), we first recall that ω_r is constructed using the unique unitary flat connection on polystable vector bundles of degree zero over X . More precisely, consider the unique unitary flat connection ∇ on a polystable vector bundle

$$E := \bigoplus_{i=1}^r L_i \in \mathcal{N}_X(r)$$

(the flat hermitian metric on E is not unique, but the flat hermitian connection is unique). Let $\tilde{\nabla}$ be the flat connection on $\text{End}(E) = E \otimes E^*$ induced by ∇ . The tangent space $T_E \mathcal{N}_X(r)$ is identified with

$$(3.20) \quad \bigoplus_{i=1}^r H^1(X, \text{End}(L_i)) = H^1(X, \mathcal{O}_X)^{\oplus r} \subset H^1(X, \text{End}(E)).$$

Using the flat unitary structure on $\text{End}(E)$, we can represent elements of $H^1(X, \text{End}(E))$ by the harmonic forms. This yields a L^2 -metric on $H^1(X, \text{End}(E))$. The restriction of this form to the subspace $H^1(X, \mathcal{O}_X)^{\oplus r}$ in (3.20) coincides with the Kähler form ω_r on $T_E \mathcal{N}_X(r)$.

The equality in (3.19) follows from the above description of ω_r . Note that the factor 2 in (3.19) appears because the map β constructed in (3.8) involves both L_i and L_i^* , and the involution of $\text{Pic}^0(X)$ defined by $L \mapsto L^*$ preserves the translation invariant Kähler form h_0 on $\text{Pic}^0(X)$.

The proposition follows from (3.18), (3.19) and (3.10). \square

There is a unique additive complex Lie group structure on X with $f^{-1}(0)$ as the identity element, where f is the map in (2.2). We fix this Lie group structure on X . The identity element $f^{-1}(0)$ will be denoted by e .

There is a natural complex group homomorphism

$$(3.21) \quad X \longrightarrow \text{Pic}^0(X)$$

defined by $x \longmapsto \mathcal{O}_X(x - e)$. This isomorphism will be useful here.

4. NON-ABELIAN THETA FUNCTIONS

In [FMN2], non-abelian theta functions on the moduli space of trivial determinant vector bundles of rank of n on the elliptic curve X were studied in terms of Weyl anti-invariant distributions in $\text{SU}(n)$. Let us recall briefly that construction, paying particular attention to the case $n = 2$, which will be especially relevant for the description of the Hilbert space associated to the quantization of the moduli space of parabolic bundles \mathcal{M}_P .

4.1. $\text{SL}_n(\mathbb{C})$ non-abelian theta functions on an elliptic curve. We start by writing the elliptic curve X in the form:

$$(4.1) \quad X = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z}), \text{ for some } \tau \in \mathbb{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}.$$

Let \mathfrak{h} be the Cartan subalgebra of $\mathfrak{sl}_n(\mathbb{C})$ consisting of diagonal matrices of trace zero, and let $\check{\Lambda}$ denote its coroot lattice. To be concrete, we identify $\mathfrak{sl}_n(\mathbb{C})$ with the space of traceless n by n complex matrices and \mathfrak{h} with the space of diagonal matrices of trace zero.

Let $\mathcal{M}_X(n)$ be the moduli space semistable vector bundles E over X of rank n with $\bigwedge^n E = \mathcal{O}_X$.

Consider the abelian variety

$$M = X \otimes \check{\Lambda} \cong \mathfrak{h}/(\check{\Lambda} \oplus \tau\check{\Lambda}).$$

The Weyl group W of $\mathfrak{sl}_n(\mathbb{C})$, given by the permutations of $\{1, \dots, n\}$, acts naturally on M , via its natural action on \mathfrak{h} . As shown in [Lo, La], the moduli space $\mathcal{M}_X(n)$ can be naturally identified with the quotient under this action

$$\mathcal{M}_X(n) = M/W \cong \mathbb{CP}^{n-1}.$$

To consider the quantization of $\mathcal{M}_X(n)$, we use the symplectic form ω induced from the symplectic structure on X and the determinant line bundle $L \longrightarrow \mathcal{M}_X(n)$ whose curvature form coincides with ω [Qu].

Let

$$p : \mathbb{C}/\mathbb{Z} \cong \mathbb{C}^* \longrightarrow \text{Pic}^0(X) \cong X$$

be the projection defined by $z + \mathbb{Z} \longmapsto z + \mathbb{Z} + \tau\mathbb{Z}$, where $z \in \mathbb{C}$. The maximal torus of diagonal matrices in $\text{SL}_n(\mathbb{C})$ will be denoted by $T_{\mathbb{C}}$ and is canonically identified with $\mathfrak{h}/\check{\Lambda}$. Let

$$(4.2) \quad q : \text{SL}_n(\mathbb{C}) \longrightarrow \text{SL}_n(\mathbb{C})/\text{SL}_n(\mathbb{C}) \cong T_{\mathbb{C}}/W$$

be the quotient map for the conjugation action of $\mathrm{SL}_n(\mathbb{C})$ on itself. We have the following commutative diagram:

$$(4.3) \quad \begin{array}{ccccccc} \mathrm{SL}_n(\mathbb{C}) & \xrightarrow{q} & T_{\mathbb{C}}/W & \longleftarrow & T_{\mathbb{C}} & = & \mathfrak{h}/\check{\Lambda} \\ & & \downarrow & & \downarrow & & \\ & & \mathcal{M}_X(n) & \longleftarrow & M & \longleftarrow & \mathfrak{h}, \end{array}$$

where the composition $\mathrm{SL}_n(\mathbb{C}) \rightarrow T_{\mathbb{C}}/W \rightarrow \mathcal{M}_X(n)$ corresponds to the Schottky map described in [FMN2].

Note that the Verlinde numbers are:

$$(4.4) \quad \dim H^0(\mathcal{M}_X(n), L^k) = \binom{n-1+k}{k}.$$

Let $\mathcal{H}(V)$ be the space of all holomorphic functions on a complex manifold V . Given a level k , the subspace of $\mathcal{H}(\mathfrak{h})$ consisting of functions θ satisfying the identity

$$(4.5) \quad \theta(v + \check{\alpha} + \tau\check{\beta}) = (e^{-2\pi i\beta(v) - \pi i\tau\langle\beta, \beta\rangle})^k \theta(v), \quad \check{\alpha}, \check{\beta} \in \check{\Lambda}$$

will be relevant for us.

Proposition 4.1 ([FMN2]). *The space of non-abelian theta functions $H^0(\mathcal{M}_X(n), L^k)$ is naturally identified with*

$$\mathcal{H}_{k,n}^+ := \{\theta \in \mathcal{H}(\mathfrak{h}) : \theta \text{ satisfies (4.5) and } w\theta = \theta, \forall w \in W\}.$$

We remark that since the quasi-periodicity condition in (4.5) does not depend on the first summand (i.e, $\check{\alpha}$) of the lattice $\check{\Lambda} \oplus \tau\check{\Lambda}$, the non-abelian theta functions in the proposition can be also considered to be Weyl invariant holomorphic functions on $T_{\mathbb{C}} = \mathfrak{h}/\check{\Lambda}$, or equivalently, Ad-invariant holomorphic functions on $\mathrm{SL}_n(\mathbb{C})$.

Motivated by the Segal–Bargmann–Hall or “coherent state” transform for Lie groups, we will now describe a way of obtaining such Ad-invariant holomorphic functions on $G = \mathrm{SL}_n(\mathbb{C})$ starting from Ad-invariant distributions on the maximal compact subgroup $K = \mathrm{SU}(n)$.

Let Λ_W^+ denote the set of dominant weights, in one to one correspondence with irreducible representations R_λ of $K = \mathrm{SU}(n)$. For $x \in K$, the expression

$$(4.6) \quad f = \sum_{\lambda \in \Lambda_W^+} \mathrm{tr}(A_\lambda R_\lambda),$$

where $A_\lambda \in \mathrm{End}(R_\lambda)$ are endomorphism-valued coefficients, defines a distribution under appropriate growth conditions on the operator norm of the A_λ (see [FMN2]).

Let $c_\lambda \geq 0$ be the eigenvalue of $-\Delta_K$, where Δ_K is the Laplace-Beltrami operator on K associated with the Ad-invariant inner product on \mathfrak{su}_n for which the roots have squared length 2, on functions of the form $\mathrm{tr}(A_\lambda R_\lambda(x))$, $A_\lambda \in \mathrm{End}(R_\lambda)$.

Given a positive parameter $t > 0$, and $\tau \in \mathbb{H}$, the (generalized) coherent state transform (CST for short) is given by associating to a distribution f as in (4.6) the holomorphic

function on $\mathrm{SL}_n(\mathbb{C})$

$$C_t f(g) := \sum_{\lambda \in \Lambda_W^+} e^{i\pi t \tau c_\lambda \mathrm{tr}(A_\lambda R_\lambda(g))}.$$

Recall from [Lo, FMN2] that non-abelian theta functions on $\mathcal{M}_X(n)$ are more conveniently described in terms of Weyl anti-invariant theta functions on \mathfrak{h} . Denote by θ_n^- the unique (up to scale) W -anti-invariant theta function of level n on \mathfrak{h} .

Let now ρ be the Weyl vector given by half the sum of the positive roots and let σ be the denominator of the Weyl character formula analytically continued to $\mathrm{SL}_n(\mathbb{C})$. Let $\check{\alpha}$ be the longest root in $\mathfrak{sl}_n(\mathbb{C})$ and let

$$D_{k,n} := \{\lambda \in \Lambda_W^+ : \langle \lambda, \check{\alpha} \rangle \leq k\}$$

be the parameter space for integrable representations of the level k affine Kac-Moody algebra $\widehat{\mathfrak{sl}_n}(\mathbb{C})_k$. Note that $\#D_{k,n} = \binom{n-1+k}{k}$, which equals the Verlinde number (4.4).

As seen in [FMN2], there is a (τ -independent) finite-dimensional space of Ad -invariant distributions $V_{k,n}$ on $\mathrm{SU}(n)$ which has an orthonormal basis labelled by the elements of $D_{k,n}$, such that the following holds:

Theorem 4.2 ([FMN2]). *Let $n > 2$ and $C^\infty(\mathrm{SU}(n))'^{\mathrm{Ad}} \supset L^2(\mathrm{SU}(n))$ denote the space of Ad -invariant distributions on $\mathrm{SU}(n)$. Restricting the CST to $V_{k,n}$, we obtain*

$$V_{k,n} \hookrightarrow C(\mathrm{SU}(n))'^{\mathrm{Ad}} \xrightarrow{C_t} \mathcal{H}(\mathrm{SL}_n(\mathbb{C}))^{\mathrm{Ad}}.$$

Moreover, the composition of maps

$$(4.7) \quad \varphi_{k,\tau} \circ C_{\frac{1}{k+n}} : V_{k,n} \longrightarrow H^0(\mathcal{M}_X(n), L^k) \subset \mathcal{H}(\mathrm{SL}_n(\mathbb{C}))^{\mathrm{Ad}},$$

where

$$(4.8) \quad \varphi_{k,\tau}(f) = e^{\frac{\|\rho\|^2}{k+n} \pi i \tau} \frac{\sigma}{\theta_n^-} f,$$

is an isomorphism; we identify a W -invariant theta function on \mathfrak{h} with an Ad -invariant function on $\mathrm{SL}_n(\mathbb{C})$ using (4.3) and (4.5).

Remark 4.3. Note that the map $\varphi_{k,\tau}$ is well defined only on $C_{\frac{1}{k+n}}(V_{k,n})$, since these holomorphic functions are divisible by θ_n^- [Lo, FMN2].

Let $\tau_2 = \mathrm{Im}(\tau) > 0$, and define the hermitian inner product on $H^0(\mathcal{M}_X(n), L^k)$ by

$$(4.9) \quad \langle \langle F_1, F_2 \rangle \rangle := \int_{q^{-1}(\mathfrak{h}_0)} \overline{F_1} F_2 |q^* \theta_n^-|^2 d\nu_{\frac{\tau_2}{k+n}}$$

(see [AdPW]), where q was defined in equation (4.2), $d\nu_{\frac{\tau_2}{k+n}}$ denotes what is known as the heat kernel measure of $\mathrm{SL}_n(\mathbb{C})$, at time $\frac{\tau_2}{k+n}$, and $\mathfrak{h}_0 \subset \mathfrak{h}$ is a fundamental domain for the action of the semi-direct product $W \ltimes (\tilde{\Lambda} \oplus \tau \tilde{\Lambda})$.

Theorem 4.4 ([FMN2]). *The map $\varphi_{k,\tau} \circ C_{\frac{1}{k+n}} : V_{k,n} \longrightarrow H^0(\mathcal{M}_X(n), L^k)$ is a unitary isomorphism.*

Let us now consider the (slightly different) case $n = 2$, which will be especially relevant in the next section, and for which the distributions in the previous theorem can be written in a simple way. For simplicity, we will state the result only for even level, which is the case we will need.

The space $V_{2k,2} \subset C(\mathrm{SU}(2))'^{Ad}$ is the k -dimensional \mathbb{C} -span of the distributions

$$(4.10) \quad \psi_{j,2k}(x) = \frac{1}{\sigma} \sum_{n \in \mathbb{Z}} (e^{2\pi i(j+2kn)x} - e^{-2\pi i(j+2kn)x}) \in C^\infty(\mathrm{SU}(2))', j = 1, \dots, k$$

(see [FMN2]).

Consider the basis of level $2k + 4$ theta functions for the elliptic curve X , namely $\{\theta_{j,2k+4}\}_{0 \leq j < 2k+4}$, with

$$\theta_{j,2k+4}(z) = \sum_{m \in \mathbb{Z}} \exp \left(\pi i \frac{\tau}{2k+4} (j + (2k+4)m)^2 + 2\pi i (j + (2k+4)m)z \right), \quad 0 \leq j < 2k+4.$$

The Weyl anti-invariant theta function of level 4 on X is given by $\theta_4^- = \theta_{1,4} - \theta_{3,4}$.

Recall from [Lo, FMN2] that the space of Weyl invariant theta functions of level $2k$ on X can be conveniently described in terms of Weyl anti-invariant theta functions of level $2k + 4$,

$$(4.11) \quad \begin{array}{ccc} H^0(X, L_0^{2k+4})^- & \cong & H^0(X, L_0^{2k})^+ \\ \theta^- & \longmapsto & \theta^+ = \theta^- / \theta_4^-, \end{array}$$

where θ_4^- is the (unique up to nonzero multiplicative constant) Weyl anti-invariant theta function of level 4 on X . The bundle of conformal blocks (of level k) over \mathcal{M}_1 , which is associated to the moduli space of semistable rank two vector bundles with trivial determinant on X , has a natural hermitian structure which is easily expressed in terms of theta functions in $H^0(X, L_0^{2k+4})^-$ as described above [AdPW, FMN2].

Theorem 4.5. *The composition of maps*

$$V_{2k,2} \xrightarrow{C_{\frac{1}{k+2}}} C_{\frac{1}{k+2}}(V_{2k,2}) \xrightarrow{\varphi_{k,\tau}} H^0(\mathcal{M}_X(2), L^{2k}) \subset \mathcal{H}(\mathrm{SL}_2(\mathbb{C}))^{Ad}$$

where $\varphi_{k,\tau}(f) = e^{\frac{1}{2k+4}\pi i \tau} \frac{\sigma}{\theta_4^-} f$, is an isomorphism. The image of the natural basis $\{\psi_{j,2k}\}_{j=1,\dots,k}$ is given by

$$(4.12) \quad \{\vartheta_{j,2k}\}_{j=1,\dots,k},$$

where $\vartheta_{j,2k} = (\theta_{j,2k+4} - \theta_{2k+4-j,2k+4}) / \theta_4^-$ [FMN2].

4.2. Non-abelian theta functions on \mathcal{M}_P . Let $\check{\Lambda}$ be the coroot lattice of $sl_2(\mathbb{C})$, and let \mathfrak{h} , as before, be the Cartan subalgebra. The abelian variety from the previous subsection is now $M = X \otimes \check{\Lambda} \cong X$.

From Proposition 2.2, we have

$$\mathcal{M}_P \cong \mathrm{Pic}^0(X)^d / \Gamma_d \cong M^d / \Gamma_d,$$

where $\Gamma_d = \mathbb{Z}_2^d \rtimes \Sigma_d$. We have the isomorphism of abelian varieties

$$M^d = (\mathfrak{h} / \check{\Lambda} \oplus \tau \check{\Lambda})^d.$$

Let $p : M^d \rightarrow M^d/\Gamma_d$ be the natural projection. From above, the pull-back of the determinant line bundle by p gives the line bundle $\bigotimes_{i=1}^d (q_i^* L_0)^2$ over M^d . Therefore, non-abelian theta functions of level k on \mathcal{M}_P will be described by Γ_d invariant products of level $2k$ theta functions on each of the factors $\mathfrak{h}/\check{\Lambda} \oplus \tau\check{\Lambda}$.

The analog of diagram (4.3) is now

$$\begin{array}{ccccccc} \mathrm{SL}_2(\mathbb{C})^d & \longrightarrow & T_{\mathbb{C}}^d/W^d & \longleftarrow & T_{\mathbb{C}}^d & = & (\mathfrak{h}/\check{\Lambda})^d \\ & & \downarrow & & \downarrow & & \\ \mathcal{M}_P & \xleftarrow{\Sigma_d} & \mathcal{M}_X(2)^d & \longleftarrow & M^d & \longleftarrow & \mathfrak{h}^d. \end{array}$$

Let K be a compact Lie group. The CST on K^d equipped with the product metric can be applied to Σ_d -invariant functions. Since the averaged heat kernel measures are the product of the d measures on each of the factors of K , we have the following commutative diagram

$$\begin{array}{ccc} (C^\infty(K^d)')^{\Sigma_d} & \hookrightarrow & C^\infty(K^d)' \\ \downarrow C_t^{\otimes d} & & \downarrow C_t^{\otimes d} \\ \mathcal{H}(G)^{\Sigma_d} & \hookrightarrow & \mathcal{H}(G). \end{array}$$

where the CST for K^d is given by

$$C_t^{\otimes d} = C_t \otimes \cdots \otimes C_t,$$

in terms of the CST C_t for K .

Definition 4.6. Let $V_k \subset C^\infty((\mathrm{SU}(2)^d)')^{\Sigma_d}$ be the vector space with basis $\{\Psi_{J,k}\}_{J=\{j_1, \dots, j_d\}, 1 \leq j_i \leq k}$, where

$$\Psi_{J,k} = \sum_{\sigma \in \Sigma_d} \psi_{j_{\sigma_1}, 2k} \otimes \cdots \otimes \psi_{j_{\sigma_d}, 2k},$$

and the distributions $\psi_{j, 2k} \in C^\infty(\mathrm{SU}(2))'$ are given in (4.10).

Let $\varphi_{k,\tau}^{\otimes d} = \varphi_{k,\tau} \otimes \cdots \otimes \varphi_{k,\tau}$ be defined on $C_{\frac{1}{k+2}}^{\otimes d}(V_k) \subset \mathcal{H}(\mathrm{SL}_2(\mathbb{C})^d)$, where $\varphi_{k,\tau}$ was defined in (4.8).

Theorem 4.7. The CST $C_{\frac{1}{k+2}}^{\otimes d}$ establishes an isomorphism between the space V_k of distributions on $\mathrm{SU}(2)^d$ and the space of non-abelian theta functions $H^0(\mathcal{M}_P, \xi^k)$ of level k , meaning the map

$$\varphi_{k,\tau}^{\otimes d} \circ C_{\frac{1}{k+2}}^{\otimes d} : V_k \longrightarrow H^0(\mathcal{M}_P, \xi^k)$$

is an isomorphism. The image of the natural basis (Definition 4.6) is given by $\{\phi_{J,k}\}_{J=\{j_1, \dots, j_d\}, 1 \leq j_i \leq k}$, where

$$\phi_{J,k}(z_1, \dots, z_d) = \sum_{\sigma \in \Sigma_d} \vartheta_{j_{\sigma_1}, 2k}(z_1) \cdots \vartheta_{j_{\sigma_d}, 2k}(z_d).$$

Proof. Recall that the determinant line bundle ξ on the moduli space of parabolic bundle \mathcal{M}_P , satisfies $\xi \cong \otimes_{i=1}^d (q_i^* L_0)^2$, where $q_i : M^d \rightarrow M$ is the projection on the i th factor. Therefore, elements in $H^0(\mathcal{M}_P, \xi^k)$ are given by Γ_d invariant theta functions of level $2k$ on M^d . From Theorem 4.5 it follows that applying $\varphi_{k,\tau}^{\otimes d} \circ C_{\frac{1}{k+2}}^{\otimes d}$ to $\Psi_{J,k}$ we get $\phi_{J,k}$. \square

Remark 4.8. We see that the Verlinde number is equal to the dimension of the space of degree k polynomials in d variables, which is consistent with $\xi \cong \mathcal{O}(1)$ on $\mathcal{M}_P \cong \mathbb{P}^d$.

Remark 4.9. Non-abelian theta functions $H^0(\mathcal{M}_P, \xi^k)$ of level k can therefore be described as the image, by a coherent state transform, of the finite dimensional space of distributions on the compact group $SU(2)^d$. In particular, following [FMN1, FMN2], in this case the CST can also be interpreted as the parallel transport of a unitary connection on the bundle of conformal blocks over $\mathcal{M}_{0,4}$.

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